

# CANCELLATIONS IN POWER SERIES OF SINE TYPE

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ABSTRACT. We present a method to study the behavior of a power series of type

$$f(x) := \sum_{n=0}^{\infty} (-1)^n c_n \frac{x^{2n+1}}{(2n+1)!}$$

when  $x \rightarrow \infty$ .

We apply our method to study the function

$$f(t) := \int_0^t \frac{dx}{x} \int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z} \{ \sin x + \sin(x-y) - \sin(x-z) - \sin(x-y+z) \}.$$

We will derive various different representations of  $f(t)$  by means of which it will be shown that  $\lim_{t \rightarrow +\infty} f(t) = 0$ , disproving a conjecture by Z. Silagadze, claiming that this limit equals  $-\pi^3/12$ .

## 1. INTRODUCTION.

In Titchmarsh [12, Section 14.32] we find two suggestive equivalents to the Riemann Hypothesis: The RH is equivalent to  $F(x) = \mathcal{O}(x^{\frac{1}{2}+\varepsilon})$ , or alternatively to  $G(x) = \mathcal{O}(x^{-\frac{1}{4}+\varepsilon})$  where

$$(1.1) \quad F(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{(n-1)! \zeta(2n)}, \quad G(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n! \zeta(2n+1)}.$$

The first equivalence is due to Riesz [8], and the second to Hardy and Littlewood [1].

Since  $\zeta(n)$  converges to 1 these series can be considered slight modifications of the exponential function. As in the case of the exponential the series are convergent everywhere but the small values they get for  $x$  large is the result of an amazing cancellation between large terms of different signs. This phenomenon happens also in the case of the sine or cosine series. In these simple cases the many algebraic properties of the corresponding sums yield the proof of the cancellations. But how can one treat a case as the series (1.1) above?

A problem in MathOverflow leads us to consider the power series

$$(1.2) \quad f(t) = \sum_{n=1}^{\infty} (-1)^n \left( \sum_{k=1}^{2n+1} \frac{H_{k-1}}{k} \right) \frac{t^{2n+1}}{(2n+1)!(2n+1)}$$

where the  $H_n = \sum_{k=1}^n \frac{1}{k}$  are the harmonic numbers. The proposer Z. Silagadze asked for a proof that  $f(t) \rightarrow -\pi^3/12$  (obtained by using some arguments from physics).

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We think that our solution presented here is interesting because it provides an example of how to treat this type of problems. It must be said that the method can be applied to the functions in (1.1), but as we will see in Section 6 it only gives another path to the connection between these series and the Riemann Hypothesis.

Silagadze's problem in MathOverflow [10] was to compute

$$(1.3) \quad \int_0^\infty \frac{dx}{x} \int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z} \{\sin x + \sin(x-y) - \sin(x-z) - \sin(x-y+z)\}$$

for which he conjectured the value  $-\pi^3/12$ .

This integral is not absolutely convergent. Being a multiple integral it is not clear in what sense he is asking to compute it. The most natural interpretation of (1.3) is to define for  $t > 0$

$$(1.4) \quad f(t) := \int_0^t \frac{dx}{x} \int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z} \{\sin x + \sin(x-y) - \sin(x-z) - \sin(x-y+z)\}.$$

and define the integral (1.3) as the  $\lim_{t \rightarrow \infty} f(t)$ . We will prove that this limit is 0.

The function  $f(t)$  defined by the triple integral in (1.4) has many interesting properties. It extends to an entire function whose power series (1.2) is a slight modification of the series for  $\sin t$ . We are interested in its behavior for  $t \rightarrow +\infty$ . This is a similar problem as the one (equivalent to the Riemann Hypothesis) introduced by M. Riesz cited above. We will solve the problem in this relatively simple case, essentially, by means of a representation of  $f(t)$  as a Fourier transform (4.5).

The transformations we apply to the function  $f(t)$  conclude with the representation as a Fourier transform in Proposition 4.4. But all our representations are needed to reach this final one which will solve Silagadze's problem.

Our formulas (2.3) and (4.5) may also be applied to the computation of the triple integral. The numerical computation of a triple integral is quite often difficult. By means of the power series or the Fourier representation it can be computed easily and above all more reliably for relatively small values of  $t$ . For large values of  $t$  we present explicitly an asymptotic expansion (Section 5) that is very well suited to compute  $f(t)$  with high precision for  $t$  large. The first term of the asymptotic expansion

$$f(t) = -\frac{\cos t \log^2 t}{2t} + \mathcal{O}\left(\frac{\log t}{t}\right)$$

shows that  $f(t)$  has zeros near the zeros of  $\cos t$  for  $t$  large.

As an application we will also obtain the x-ray of  $f(t)$ . It shows that the zeros with small absolute value are real. On average there are two zeros on each interval of length  $2\pi$ . The first zeros are separated by approximately 4, 2, 4, 2, ... The x-ray also shows that the general properties of  $f(t)$  are very similar to those of  $\sin t$ .

To finish the paper we give some details about the Riesz function analogous to  $f(t)$ , also explaining why we may not hope that an analogous analysis will solve the RH. As expected!

## 2. THE ENTIRE FUNCTION $f(t)$ .

**Proposition 2.1.** *The function  $f(t)$  defined for  $t > 0$  by the absolutely convergent integral*

$$(2.2) \quad f(t) := \int_0^t \frac{dx}{x} \int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z} \{\sin x + \sin(x-y) - \sin(x-z) - \sin(x-y+z)\}$$

extends to an entire function with power series expansion

$$(2.3) \quad f(t) = \sum_{n=1}^{\infty} (-1)^n \left( \sum_{k=1}^{2n+1} \frac{H_{k-1}}{k} \right) \frac{t^{2n+1}}{(2n+1)!(2n+1)}$$

where  $H_n = \sum_{k=1}^n \frac{1}{k}$  is the  $n$ -th harmonic number.

*Proof.* Applying the Mean Value Theorem we get the following bound of the absolute value of the integrand in the definition of  $f(t)$

$$\frac{1}{x} \cdot \frac{1}{y} \left| \frac{\sin x - \sin(x-z)}{z} - \frac{\sin(x-y+z) - \sin(x-y)}{z} \right| \leq \frac{2}{xy}.$$

When integrating over  $z \in (0, y)$  we get something bounded by  $2/x$ , and integrating this over  $y \in (0, x)$  we get something bounded by 2. The integral of this over  $x \in (0, t)$  yields something bounded by  $2t$ . This shows that the integral in (2.2) is absolutely convergent.

Changing variables  $x = tu$ ,  $y = tv$ ,  $z = tw$  yields

$$f(t) = \int_0^1 \frac{du}{u} \int_0^u \frac{dv}{v} \int_0^v \frac{dw}{w} \{ \sin(tu) + \sin(tu-tv) - \sin(tu-tw) - \sin(tu-tv+tw) \}$$

and renaming the variables

$$f(t) = \int_0^1 \frac{dx}{x} \int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z} \{ \sin(tx) + \sin(tx-ty) - \sin(tx-tz) - \sin(tx-ty+tz) \}.$$

This representation shows that  $f(t)$  extends to an entire function with power series expansion

$$f(t) = \sum_{n=0}^{\infty} (-1)^n A_n \frac{t^{2n+1}}{(2n+1)!}$$

where

$$A_n = \int_0^1 \frac{dx}{x} \int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z} \{ x^{2n+1} + (x-y)^{2n+1} - (x-z)^{2n+1} - (x-y+z)^{2n+1} \}.$$

Notice that  $A_0 = 0$ . To compute the other  $A_n$  we get successively

$$\begin{aligned} \int_0^y \{ x^{2n+1} - (x-z)^{2n+1} \} \frac{dz}{z} &= - \sum_{k=1}^{2n+1} (-1)^k \binom{2n+1}{k} \frac{y^k}{k} x^{2n+1-k}, \\ \int_0^x \frac{dy}{y} \int_0^y \{ x^{2n+1} - (x-z)^{2n+1} \} \frac{dz}{z} &= - \sum_{k=1}^{2n+1} (-1)^k \binom{2n+1}{k} \frac{x^{2n+1}}{k^2}, \\ \int_0^1 \frac{dx}{x} \int_0^x \frac{dy}{y} \int_0^y \{ x^{2n+1} - (x-z)^{2n+1} \} \frac{dz}{z} &= - \frac{1}{2n+1} \sum_{k=1}^{2n+1} (-1)^k \binom{2n+1}{k} \frac{1}{k^2} \\ &= \frac{1}{2n+1} \sum_{k=1}^{2n+1} \frac{H_k}{k} \end{aligned}$$

where the last step is proved in Lemma 2.4.

For the other part of  $A_n$  we compute

$$\begin{aligned} \int_0^y \{(x-y)^{2n+1} - (x-y+z)^{2n+1}\} \frac{dz}{z} &= - \sum_{k=1}^{2n+1} \binom{2n+1}{k} \frac{y^k}{k} (x-y)^{2n+1-k}, \\ \int_0^x \frac{dy}{y} \int_0^y \{(x-y)^{2n+1} - (x-y+z)^{2n+1}\} \frac{dz}{z} \\ &= - \sum_{k=1}^{2n+1} \frac{1}{k} \binom{2n+1}{k} \int_0^x y^k (x-y)^{2n+1-k} \frac{dy}{y}. \end{aligned}$$

In the last integral we change variables  $y = xu$  yielding

$$\begin{aligned} &= - \sum_{k=1}^{2n+1} \frac{x^{2n+1}}{k} \binom{2n+1}{k} \int_0^1 u^k (1-u)^{2n+1-k} \frac{du}{u} = \\ &= - \sum_{k=1}^{2n+1} \frac{x^{2n+1}}{k} \frac{(2n+1)!}{k!(2n+1-k)!} \frac{(k-1)!(2n+1-k)!}{(2n+1)!} = - \sum_{k=1}^{2n+1} \frac{x^{2n+1}}{k^2} \end{aligned}$$

so that

$$\int_0^1 \frac{dx}{x} \int_0^x \frac{dy}{y} \int_0^y \{(x-y)^{2n+1} - (x-y+z)^{2n+1}\} \frac{dz}{z} = - \frac{1}{2n+1} \sum_{k=1}^{2n+1} \frac{1}{k^2}.$$

Collecting all our partial results we obtain

$$A_n = \frac{1}{2n+1} \sum_{k=1}^{2n+1} \frac{H_k}{k} - \frac{1}{2n+1} \sum_{k=1}^{2n+1} \frac{1}{k^2} = \frac{1}{2n+1} \sum_{k=1}^{2n+1} \frac{H_{k-1}}{k}.$$

□

**Lemma 2.4.** *For any natural number  $n$  we have*

$$(2.5) \quad \sum_{k=1}^n (-1)^k \binom{n}{k} \frac{1}{k^2} = - \sum_{k=1}^n \frac{H_k}{k}.$$

*Proof.* To prove  $a_n = b_n$  it is sufficient to prove  $a_1 = b_1$  and  $a_n - a_{n-1} = b_n - b_{n-1}$ . In our case the equality for  $n = 1$  is checked easily. So, we have to show

$$\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \frac{1}{k^2} - \sum_{k=1}^{n-1} (-1)^{k+1} \binom{n-1}{k} \frac{1}{k^2} = \frac{H_n}{n}.$$

Since  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ , the equality is equivalent to

$$\frac{(-1)^{n+1}}{n^2} + \sum_{k=1}^{n-1} (-1)^{k+1} \binom{n-1}{k-1} \frac{1}{k^2} = \frac{H_n}{n}$$

or, multiplying by  $n$

$$(2.6) \quad \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \frac{1}{k} = H_n.$$

This is formula 0.155.4 in Gradshteyn and Ryzhik [2, p. 4].

□

### 3. PLOT OF THE FUNCTION $f(t)$ .

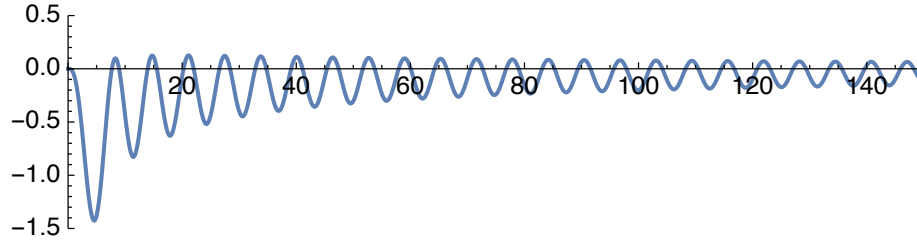
The power series expansion of  $f(t)$  is analogous to the one for  $\sin t$ . These power series are not well suited for computation, because of the violent cancellation between large terms. Nevertheless, one may use it using high precision in the computation of the terms to get approximate values. Assuming  $t > 0$ , each term of the power series (2.3) is in absolute value less than  $t^{2n+1}/(2n+1)!$ . Therefore the error committed by summing only the first  $N > 3t$  terms of the series (2.3) is less than  $2^{-2N}$ . It follows that to compute  $f(t)$  with error less than  $2\varepsilon$  we need only compute the sum of the first  $N$  terms of the series with error less than  $\varepsilon$ , taking  $N$  large enough so that

$$N \geq \frac{\log(1/\varepsilon)}{2 \log 2}, \quad N > 3t.$$

All the terms of the series are less than  $t^{2n+1}/(2n+1)! < (\frac{et}{2n+1})^{2n+1} \leq e^t$ . So, we must compute each term with an error less than  $\varepsilon/N$ , for which it will suffice to compute each term working with a precision

$$P := \frac{t + \log(N/e)}{\log 10} \quad \text{decimal digits.}$$

Of course this will be difficult for  $t$  very large, but today we may easily compute with thousands of digits of precision.



We may say that the computation of the power series is much easier than the computation of the multiple integral (2.2) or any other multiple integral giving  $f(t)$  considered in this paper. This is true even when  $t$  is small.

### 4. COMPUTATION OF THE LIMIT OF $f(t)$ .

First we prove the following integral representation of  $f(t)$ :

**Proposition 4.1.** *For  $t > 0$  we have*

$$(4.2) \quad f(t) = \int_0^t du \int_0^1 \int_0^1 \left( \frac{x \sin u}{u(1-x)(1-xy)} - \frac{\sin(ux)}{u(1-x)(1-y)} + \frac{\sin(uxy)}{u(1-y)(1-xy)} \right) dx dy.$$

*Proof.* Starting from the power series expansion it is easy to get

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} (-1)^n \sum_{k=1}^{2n+1} \frac{1}{k} \sum_{j=1}^{k-1} \frac{1}{j} \frac{t^{2n+1}}{(2n+1)!(2n+1)} \\ &= \int_0^t du \int_0^1 dx \int_0^1 dy \sum_{n=1}^{\infty} (-1)^n \sum_{k=1}^{2n+1} x^{k-1} \sum_{j=1}^{k-1} y^{j-1} \frac{u^{2n}}{(2n+1)!}. \end{aligned}$$

If  $C$  is a circle with radius  $r > t$ , we may express the factorial by means of the Residues Theorem

(4.3)

$$f(t) = \int_0^t du \int_0^1 dx \int_0^1 dy \frac{1}{2\pi i} \int_C \frac{e^z}{z^2} \sum_{n=1}^{\infty} (-1)^n \sum_{k=1}^{2n+1} x^{k-1} \sum_{j=1}^{k-1} y^{j-1} (u/z)^{2n} dz.$$

(Since  $|u/z| < 1$  all these series converge absolutely and the order of the integrals and summations may be interchanged). This can be written as

$$f(t) = \int_0^t du \int_0^1 dx \int_0^1 dy \frac{1}{2\pi i} \int_C \frac{e^z}{z^2} \frac{u^4 x^3 y z^2 - u^2 x z^4 - u^2 x^2 z^4 - u^2 x^2 y z^4}{(u^2 + z^2)(u^2 x^2 + z^2)(u^2 x^2 y^2 + z^2)} dz.$$

By the Residue Theorem we obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_C \frac{e^z}{z^2} \frac{u^4 x^3 y z^2 - u^2 x z^4 - u^2 x^2 z^4 - u^2 x^2 y z^4}{(u^2 + z^2)(u^2 x^2 + z^2)(u^2 x^2 y^2 + z^2)} dz \\ = \frac{x \sin u}{u(1-x)(1-xy)} - \frac{\sin(ux)}{u(1-x)(1-y)} + \frac{\sin(uxy)}{u(1-y)(1-xy)} \end{aligned}$$

establishing our claim (4.2). Observe that since our integral representation (4.2) appears after computing one of the integrals in the absolutely convergent integral (4.3), the integrals in our new representation are also absolutely convergent.  $\square$

Notice that in the representation (4.2) we may interchange the integral in  $u$  with the integral in  $(x, y)$ , but then it is not easy to justify the interchange of the limit in  $t$  with the integrals, because the integrand is not dominated by an integrable function. If we would proceed formally in this way we would easily get that the limit is 0, but, as said before, this is not allowed. Therefore we follow another path.

**Proposition 4.4.** *For any complex  $t$  we have*

$$(4.5) \quad f(t) = \int_0^1 \left( \frac{1}{2} \log^2(1-x) + \sum_{n=1}^{\infty} \frac{(1-x)^n - 1}{n^2} \right) \frac{\sin(tx)}{x} dx.$$

*Proof.* We take  $t > 0$  in the representation (4.2), and change the order of integration

$$\begin{aligned} f(t) = \\ \int_0^1 \int_0^1 dx dy \int_0^t \left( \frac{x \sin u}{u(1-x)(1-xy)} - \frac{\sin(ux)}{u(1-x)(1-y)} + \frac{\sin(uxy)}{u(1-y)(1-xy)} \right) du. \end{aligned}$$

Now subdivide the inner integral in three and change variables appropriately

$$\begin{aligned} f(t) = \int_0^1 \int_0^1 dx dy \left( \int_0^t \frac{x \sin u}{u(1-x)(1-xy)} du \right. \\ \left. - \int_0^{tx} \frac{\sin u}{u(1-x)(1-y)} du + \int_0^{txy} \frac{\sin u}{u(1-y)(1-xy)} du \right). \end{aligned}$$

We use Iverson's notation [5], so that, for any proposition  $P$ , the symbol  $[P]$  is 1 if  $P$  is true and 0 if it is false. In this way we may write

$$\begin{aligned} f(t) = \\ \int_0^1 \int_0^1 dx dy \int_0^t \left( \frac{x[u < t] \sin u}{u(1-x)(1-xy)} du - \frac{[u < tx] \sin u}{u(1-x)(1-y)} + \frac{[u < txy] \sin u}{u(1-y)(1-xy)} \right) du. \end{aligned}$$

To simplify the notation we put  $a := u/t$ , so that always  $0 < a < 1$ . Interchanging the integrals we obtain

$$f(t) = \int_0^t \frac{\sin u}{u} du \int_0^1 \int_0^1 \left( \frac{x}{(1-x)(1-xy)} du - \frac{[a < x]}{(1-x)(1-y)} + \frac{[a < xy]}{(1-y)(1-xy)} \right) dx dy$$

and will compute the inner double integral in  $x$  and  $y$ . This is

$$J(a) := \int_0^1 \int_0^1 \left( \frac{x}{(1-x)(1-xy)} du - \frac{[a < x]}{(1-x)(1-y)} + \frac{[a < xy]}{(1-y)(1-xy)} \right) dx dy.$$

We subdivide the square  $(0, 1)^2$  into three disjoint sets

$$S_1 := \{(x, y) : 0 < x \leq a, 0 < y < 1\}, \quad S_2 := \{(x, y) : a < x < 1, 0 < xy \leq a\}, \\ S_3 := \{(x, y) : a < xy\}.$$

In  $S_1$  the integrand is  $= \frac{x}{(1-x)(1-xy)}$ , and in  $S_2$  it is

$$= \frac{x}{(1-x)(1-xy)} - \frac{1}{(1-x)(1-y)} = -\frac{1}{(1-y)(1-xy)}.$$

In  $S_3$  the integrand is equal to

$$= \frac{x}{(1-x)(1-xy)} - \frac{1}{(1-x)(1-y)} + \frac{1}{(1-y)(1-xy)} = 0$$

so that

$$J(a) = \int_0^1 dy \int_0^a \frac{x}{(1-x)(1-xy)} dx - \int_a^1 dx \int_0^{a/x} \frac{1}{(1-y)(1-xy)} dy \\ = \frac{1}{2} \log^2(1-a) - \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{(1-a)^n}{n^2}.$$

This yields

$$f(t) = \int_0^t \left( \frac{1}{2} \log^2(1-u/t) + \sum_{n=1}^{\infty} \frac{(1-u/t)^n - 1}{n^2} \right) \frac{\sin u}{u} du.$$

A change of variables  $u = tx$  yields (4.5) for  $t > 0$ , and it is clear that the two sides are entire functions.  $\square$

**Proposition 4.6.** *We have*

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow +\infty} \int_0^1 \left( \frac{1}{2} \log^2(1-x) + \sum_{n=1}^{\infty} \frac{(1-x)^n - 1}{n^2} \right) \frac{\sin(tx)}{x} dx = 0.$$

*Proof.* This is an example of the Riemann-Lebesgue Lemma, once it has been shown that the function

$$g(x) := \frac{1}{x} \left( \frac{1}{2} \log^2(1-x) + \sum_{n=1}^{\infty} \frac{(1-x)^n - 1}{n^2} \right)$$

is in  $\mathcal{L}^1[0, 1]$ . This is a simple exercise.  $\square$

## 5. ASYMPTOTIC EXPANSION.

The representation (4.4) as a Fourier integral yields by known methods an asymptotic expansion for  $f(t)$ . First notice that the dilogarithm function  $\text{Li}_2(z)$  can be defined in the cut plane  $\mathbb{C} \setminus [1, \infty)$  by

$$(5.1) \quad \text{Li}_2(z) = - \int_0^z \log(1-u) \frac{du}{u}$$

where the path of integration is the segment joining 0 and  $z$ . For  $z$  in the plane with two cuts along the real axis  $(-\infty, 0]$  and  $[1, +\infty)$  the dilogarithm satisfies the Euler functional equation (cf. Lewin [6, (1.12), p. 5])

$$(5.2) \quad \text{Li}_2(z) + \text{Li}_2(1-z) = \frac{\pi^2}{6} - \log z \log(1-z).$$

Here and in the sequel we will denote by  $\log w$  the principal branch of the logarithm.

**Definition 5.3.** Let  $\Omega$  be the complex plane with the two cuts  $(-\infty, 0]$  and  $[1, +\infty)$ . We define the function  $g(z)$  holomorphic in  $\Omega$  by

$$(5.4) \quad g(z) := \left( \frac{1}{2} \log^2(1-z) - \frac{\pi^2}{6} + \text{Li}_2(1-z) \right) \frac{1}{z} \\ = \left( \frac{1}{2} \log^2(1-z) - \text{Li}_2(z) - \log z \log(1-z) \right) \frac{1}{z}.$$

It is easier to give explicitly the asymptotic expansion of the Fourier transform

$$(5.5) \quad J(t) := \int_0^1 g(x) e^{ixt} dx.$$

Since  $f(t) = \text{Im } J(t)$  its asymptotic expansion can be obtained easily from the one of  $J(t)$ .

**Proposition 5.6.** For any  $t > 0$  we have  $f(t) = \text{Im } J(t)$  where

$$(5.7) \quad J(t) = i \int_0^\infty g(iy) e^{-ty} dy - i e^{it} \int_0^\infty g(1+iy) e^{-ty} dy.$$

*Proof.* By the definition of  $g(z)$  and (4.4) we have

$$f(t) = \text{Im} \int_0^1 g(x) e^{ixt} dx.$$

We apply Cauchy's Theorem to a rectangle with vertices at 0, 1,  $1+iR$ ,  $iR$  with  $R > 0$  and let  $R \rightarrow +\infty$ . In this way the integral in  $(0, 1)$  can be converted into the two infinite integrals in the Proposition. The bounds are easy.  $\square$

Since the two integrals in Proposition 5.6 are Laplace integrals we may apply Watson's Lemma [3, 17.03, p. 501] to get their asymptotic expansions. In this case we have the additional difficulty of logarithmic singularities at the extremes of the integrals at 0 and 1. We follow the path in Lyness [7]. To get the asymptotic expansion we need the behavior of  $g(z)$  near  $z = 0$  and  $z = 1$ .



**Lemma 5.8.** *For  $|z| < 1$  in  $\Omega$  we have*

$$(5.9) \quad g(z) = \sum_{n=0}^{\infty} \left( \frac{H_{n+1}}{n+1} - \frac{2}{(n+1)^2} \right) z^n + \left( \sum_{n=0}^{\infty} \frac{z^n}{n+1} \right) \log z \\ := \sum_{n=0}^{\infty} A_n z^n + \left( \sum_{n=0}^{\infty} B_n z^n \right) \log z.$$

For  $z \in \Omega$ , with  $|1-z| < 1$  we have

$$(5.10) \quad g(z) = - \sum_{n=0}^{\infty} \Psi'(n+1)(1-z)^n + \frac{1}{2} \left( \sum_{n=0}^{\infty} (1-z)^n \right) \log^2(1-z) \\ := \sum_{n=0}^{\infty} C_n (1-z)^n + \left( \sum_{n=0}^{\infty} D_n (1-z)^n \right) \log^2(1-z)$$

where  $\Psi(z) = \Gamma'(z)/\Gamma(z)$  is the digamma function.

*Proof.* Near  $z = 0$  we have  $g(z) = g_1(z) + g_2(z) \log z$  where by (5.4) we have

$$g_1(z) = \left( \frac{1}{2} \log^2(1-z) - \text{Li}_2(z) \right) \frac{1}{z}, \quad g_2(z) = -\frac{\log(1-z)}{z}.$$

$g_1(z)$  and  $g_2(z)$  are holomorphic at  $z = 0$ . Expanding in power series at  $z = 0$  we get (5.9).

Near  $z = 1$  we have  $g(z) = g_3(z) + g_4(z) \log^2(1-z)$  with  $g_3(z)$  and  $g_4(z)$  holomorphic at  $z = 1$ . By (5.4)

$$g_3(z) = -\frac{\pi^2}{6z} + \frac{\text{Li}_2(1-z)}{z}, \quad g_4(z) = \frac{1}{2z}.$$

The functions  $g_3(z)$  and  $g_4(z)$  are holomorphic at  $z = 1$  and expanding them in power series we obtain (5.10). We notice the equalities.

$$(5.11) \quad \Psi(n+1) = H_n - \gamma, \quad \Psi'(n+1) = \frac{\pi^2}{6} - \sum_{k=1}^n \frac{1}{k^2}.$$

□

**Proposition 5.12.** *The following asymptotic expansion is valid for  $t \rightarrow +\infty$*

$$(5.13) \quad J(t) \sim \sum_{n=0}^{\infty} i^{n+1} A_n \frac{n!}{t^{n+1}} + \sum_{n=0}^{\infty} i^{n+1} B_n \left( \Psi(n+1) - \log t + \frac{\pi i}{2} \right) \frac{n!}{t^{n+1}} \\ - i e^{it} \sum_{n=0}^{\infty} (-i)^n C_n \frac{n!}{t^{n+1}} - i e^{it} \sum_{n=0}^{\infty} (-i)^n D_n \left\{ \Psi'(n+1) + \left( \gamma - H_n + \log t + \frac{\pi i}{2} \right)^2 \right\} \frac{n!}{t^{n+1}}$$

where the coefficients  $A_n$ ,  $B_n$ ,  $C_n$  and  $D_n$  are defined in Lemma 5.8.

*Proof.* By the results in Lyness [7] the asymptotic expansion of  $J(t)$  is obtained by integrating term by term the expression (5.7) after substituting the power series at

the extremes at 0 and 1, respectively. Explicitly

$$\begin{aligned} J(t) &\sim \sum_{n=0}^{\infty} i^{n+1} A_n \int_0^{\infty} y^n e^{-ty} dy + \sum_{n=0}^{\infty} i^{n+1} B_n \int_0^{\infty} y^n \left( \log y + \frac{\pi i}{2} \right) e^{-ty} dy \\ &- i e^{it} \sum_{n=0}^{\infty} (-i)^n C_n \int_0^{\infty} y^n e^{-ty} dy - i e^{it} \sum_{n=0}^{\infty} (-i)^n D_n \int_0^{\infty} y^n \left( \log y - \frac{\pi i}{2} \right)^2 e^{-ty} dy \end{aligned}$$

Computing the integrals we obtain (5.13).  $\square$

**Corollary 5.14.** *The first order terms of the asymptotic expansion of  $f(t)$  are*

$$(5.15) \quad -\frac{\cos t \log^2 t}{2} \frac{1}{t} + \left( \frac{\pi \sin t}{2} - \gamma \cos t - 1 \right) \frac{\log t}{t} + \left( \frac{5\pi^2 \cos t}{24} - \frac{\gamma^2 \cos t}{2} + \frac{\gamma \pi \sin t}{2} - 1 - \gamma \right) \frac{1}{t}$$

where  $\gamma$  is Euler's constant.

## 6. RIESZ TYPE FUNCTIONS

We have proved that the function  $f(t)$  given by the power series (1.2) tends to 0 by writing it as a Fourier transform of an  $L^1$  function. As an example of what happens in the case of a Riesz-type function we just consider Riesz's  $F(x)$ . Riesz proves [8]

$$(6.1) \quad F(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{(n-1)! \zeta(2n)} = x \sum_{n=2}^{\infty} \frac{\mu(n)}{n^2} e^{-\frac{x}{n^2}}.$$

The Riemann Hypothesis is equivalent to  $F(x) = \mathcal{O}(x^{\frac{1}{4}+\varepsilon})$  for any  $\varepsilon > 0$ . Riesz shows that  $F(x)$  is not  $\mathcal{O}(x^\alpha)$  for any  $\alpha < 1/4$ . Therefore, it is not true that  $F(x)$  converges to 0 when  $x \rightarrow \infty$ . Riesz proves that  $F(x)x^{-1/2}$  converges to 0. With some work we may reprove this. Noticing that

$$(6.2) \quad \sqrt{t} e^{-t} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{\cos(\frac{3}{2} \arctan x)}{(1+x^2)^{3/4}} \cos(xt) dx, \quad t > 0$$

we can prove

$$(6.3) \quad \frac{F(x)}{\sqrt{x}} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \left( \sum_{n=1}^{\infty} \mu(n) \frac{n \cos(\frac{3}{2} \arctan(n^2 t))}{(1+n^4 t^2)^{3/4}} \right) \cos(xt) dt.$$

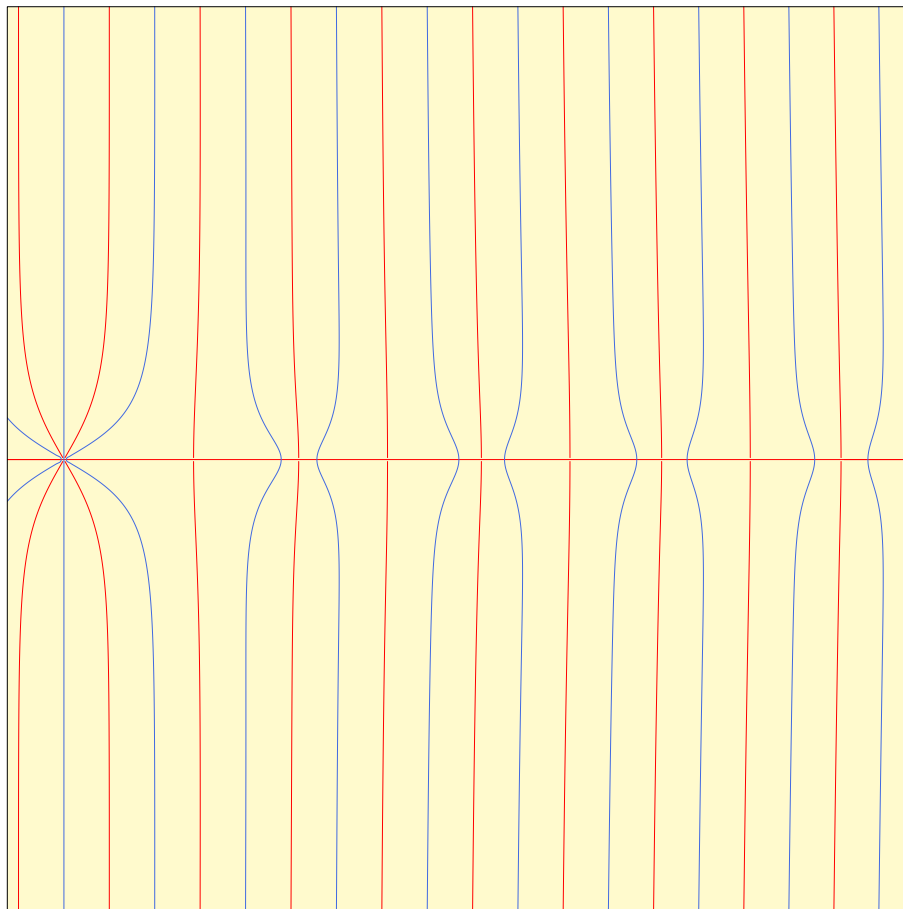
We apply summation by parts and use  $M(x) := \sum_{n \leq x} \mu(n) = \mathcal{O}(xe^{-c\sqrt{\log x}})$  to show that

$$(6.4) \quad g(t) := \sum_{n=1}^{\infty} \mu(n) \frac{n \cos(\frac{3}{2} \arctan(n^2 t))}{(1+n^4 t^2)^{3/4}}$$

is in  $L^1(0, \infty)$ . This implies that  $F(x)x^{-1/2} = o(1)$ .

We do not give the details of the proofs, but they are clearly more complicated than those given by Riesz.

We can also obtain in this way that  $M(x) = \mathcal{O}(x^a)$  with  $\frac{1}{2} \leq a < 1$  would imply that  $\zeta(s)$  does not vanish on  $\operatorname{Re} s > a$ . Again quite a difficult way to get this simple result.

7. THE X-RAY OF THE FUNCTION  $f(t)$ .

This represents  $f(z)$  on the square  $(-2, 30) \times (-16, 16)$ . In red lines where  $f(z)$  takes real values, in blue lines where  $f(z)$  is purely imaginary. At  $t = 0$  we observe a triple zero. Table 1 contains some more zeros of  $f(t)$ .

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TABLE 1. First zeros of  $f(t)$ .

1	7.67705050991057	16	53.7503680917393	31	101.945883830959
2	8.93841793140833	17	57.9542719238923	32	104.179794651601
3	13.9549229413771	18	60.0643806701981	33	108.230057788273
4	15.5679745247884	19	64.2391371546902	34	110.475488483408
5	20.2405336746206	20	66.3740436665322	35	114.514173951783
6	22.0185470304095	21	70.5238571367734	36	116.770227743121
7	26.5267465905312	22	72.6803262355070	37	120.798238189933
8	28.4064753351502	23	76.8084524955294	38	123.064133569705
9	32.8127796622118	24	78.9839168600548	39	127.082255551763
10	34.7636137360244	25	83.0929401032590	40	129.357306301363
11	39.0985258618221	26	85.2853203526836	41	133.366230408727
12	41.1028595266668	27	89.3773338407949	42	135.649829884233
13	45.3839993880412	28	91.5849167023848	43	139.650166567860
14	47.4305876227269	29	95.6616452128959	44	141.941775186108
15	51.6692361998946	30	97.8829983472283	45	145.934067362779

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